

Association schemes related to Delsarte-Goethals codes ^{*}

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Abstract

In this paper, we construct an infinite series of 9-class association schemes from a refinement of the partition of Delsarte-Goethals codes by their Lee weights. The explicit expressions of the dual schemes are determined through direct manipulations of complicated exponential sums. As a byproduct, the other three infinite families of association schemes are also obtained as fusion schemes and quotient schemes.

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1 Introduction

Since the discovery of the \mathbb{Z}_4 -linearity of Kerdock, Preparata, Goethals, and Delsarte-Goethals codes (see [10]), there have been several applications of \mathbb{Z}_4 -linear codes in the constructions of combinatorial configurations, such as t -designs

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and association schemes. According to the paper [21] of Solé, the discovery of the \mathbb{Z}_4 -linearity is motivated by a construction of association schemes due to Liebler and Mena (see [17]) using Galois rings of characteristic 4. The research on constructing t -designs from linear codes over \mathbb{Z}_4 is first put forwarded by Harada in [11], and subsequently followed by Helleseth and his collaborators in a series of papers (see [20] and the references therein).

Association schemes form a central part of algebraic combinatorics, and have played important roles in several branches of mathematics, such as coding theory and graph theory. Henry Cohn and his collaborators in [2] conjectured that a certain 3-class association scheme on 64 points determines a universally optimal configuration in \mathbb{R}^{14} . Then in [1] Abdukhalikov, Bannai and Suda generalized this example in terms of binary and quaternary Kerdock and Preparata codes, as well as in terms of maximal set of mutually unbiased basis. To be specific, they constructed a series of 3-class association schemes using the partition of shortened Kerdock codes induced by their Lee weights, and dual schemes on the cosets of punctured Preparata codes. This motivates us to look at another important class of quaternary codes, the Delsarte-Goethals (\mathcal{DG}) codes.

The \mathcal{DG} code has 6 nonzero Lee weights. It turns out that the partition of the \mathcal{DG} code by its Lee weights does not give an association scheme. We should further pick out the codewords of the Kerdock code, and this eventually yields a 9-class association scheme. Using complicated exponential sums and heavy computations, we get the structure of the dual scheme and their eigenmatrices. The dual scheme can not be obtained in an analogous way on the cosets of the Goethals code. There is a conjectured 22-class association scheme on the \mathcal{DG} code by the partition using its complete weight enumerator, but our scheme is not a fusion scheme of this conjectured scheme.

When $m = 3$, the image of the \mathcal{DG} code under Gray map is linear, and we have checked that the image of our 9-class association scheme under Gray map remains to be a scheme but with different parameters. When $m > 3$, the Gray map image of the \mathcal{DG} code is no longer nonlinear. It is not clear whether we can find translate schemes in elementary abelian 2-groups with the same parameters as the schemes we construct in this paper.

This paper is organized as follows. In Section 2, we give some preliminaries on association schemes, Galois rings, and some quaternary codes with their Lee weight distributions. In Section 3, we describe our construction of a 9-class association scheme coming from refining the partition of the \mathcal{DG} code by its Lee weights. Its dual scheme and eigenmatrices are explicitly determined. Also a 7-class scheme as its fusion scheme and a 5-class scheme as its quotient scheme are obtained. The eigenmatrices of these schemes are listed in Appendix A. The proof of our main result is provided in Section 4.

2 Preliminaries

2.1 Association schemes.

Let X be a nonempty finite set, and a set of symmetric relations R_0, R_1, \dots, R_d be a partition of $X \times X$ such that $R_0 = \{(x, x) | x \in X\}$. Denote by A_i the adjacency matrix of R_i for each i , whose (x, y) -th entry is 1 if $(x, y) \in R_i$ and 0 otherwise. We call $(X, \{R_i\}_{i=0}^d)$ a *d-class association scheme* if there exist numbers $p_{i,j}^k$ such that

$$A_i A_j = \sum_{k=0}^d p_{i,j}^k A_k.$$

These numbers are called the intersection numbers of the scheme. The \mathbb{C} -linear span of A_0, A_1, \dots, A_d forms a semisimple algebra of dimension $d+1$, called the *Bose-Mesner algebra* of the scheme. With respect to the basis A_0, A_1, \dots, A_d , the matrix of the multiplication by A_i is denoted by B_i , namely

$$A_i(A_0, A_1, \dots, A_d) = (A_0, A_1, \dots, A_d)B_i, \quad 0 \leq i \leq d.$$

Since each A_i is symmetric, this algebra is commutative. There exists a set of minimal idempotents E_0, E_1, \dots, E_d which also forms a basis of the algebra. The $(d+1) \times (d+1)$ matrix P such that

$$(A_0, A_1, \dots, A_d) = (E_0, E_1, \dots, E_d)P$$

is called the *first eigenmatrix* of the scheme. Dually, the $(d+1) \times (d+1)$ matrix Q such that

$$(E_0, E_1, \dots, E_d) = \frac{1}{|X|}(A_0, A_1, \dots, A_d)Q$$

is called the *second eigenmatrix* of the scheme. Clearly, we have $PQ = |X|I$.

We call an association scheme $(X, \{R_i\}_{i=0}^d)$ a *translation association scheme* or a *Schur ring* if X is a (additively written) finite abelian group and there exists a partition $S_0 = \{0\}, S_1, \dots, S_d$ of X such that

$$R_i = \{(x, x+y) | x \in X, y \in S_i\}.$$

For brevity, we will just say that $(X, \{S_i\}_{i=0}^d)$ is an association scheme.

Assume that $(X, \{S_i\}_{i=0}^d)$ is a translation association scheme. There is an equivalence relation defined on the character group \hat{X} of X as follows: $\chi \sim \chi'$ if and only if $\chi(S_i) = \chi'(S_i)$ for each $0 \leq i \leq d$. Here $\chi(S) = \sum_{g \in S} \chi(g)$, for any $\chi \in \hat{X}$, and $S \subseteq X$. Denote by D_0, D_1, \dots, D_d the equivalence classes, with D_0 consisting of only the principal character. Then $(\hat{X}, \{D_i\}_{i=0}^d)$ forms a translation

association scheme, called the *dual* of $(X, \{S_i\}_{i=0}^d)$. The first eigenmatrix of the dual scheme is equal to the second eigenmatrix of the original scheme. Please refer to [4] and [7] for more details.

We shall need the following well-known criterion due to Bannai [3] and Muzychuk [18], called the *Bannai-Muzychuk criterion*: *Let P be the first eigenmatrix of an association scheme $(X, \{R_i\}_{0 \leq i \leq d})$, and $\Lambda_0 := \{0\}, \Lambda_1, \dots, \Lambda_{d'}$ be a partition of $\{0, 1, \dots, d\}$. Then $(X, \{R_{\Lambda_i}\}_{0 \leq i \leq d'})$ forms an association scheme if and only if there exists a partition $\{\Delta_i\}_{0 \leq i \leq d'}$ of $\{0, 1, 2, \dots, d\}$ with $\Delta_0 = \{0\}$ such that each (Δ_i, Λ_j) -block of P has a constant row sum. Moreover, the constant row sum of the (Δ_i, Λ_j) -block is the (i, j) -th entry of the first eigenmatrix of the fusion scheme.*

2.2 Quaternary codes

Let $\mu : \mathbb{Z}_4 \rightarrow \mathbb{Z}_2$ denote the modulo 2 reduction map. We extend μ to $\mathbb{Z}_4[x]$ in the natural way. A monic polynomial $g(x) \in \mathbb{Z}_4[x]$ is said to be basic irreducible if $\mu(g(x))$ is a monic irreducible polynomial in $\mathbb{Z}_2[x]$. A Galois ring $R = GR(4, m)$ of characteristic 4 with 4^m elements is defined as the quotient ring $\mathbb{Z}_4[x]/(f(x))$, where $f(x)$ is a monic basic irreducible polynomial of degree m . The collection of non-units of R forms the unique maximal ideal $2R$, so R is a local ring. Clearly, μ has a natural extension to $R[x]$ and $\mu(R) = R/2R$ is isomorphic to a finite field \mathbb{F}_q of size $q = 2^m$.

As a multiplicative group, the units R^* of R has a cyclic subgroup of order $2^m - 1$, whose generator is denoted by β . Let $\mathcal{T} = \{0, 1, \beta, \dots, \beta^{2^m-2}\}$. Every element $z \in R$ can be expressed uniquely as

$$z = A + 2B, \quad A, B \in \mathcal{T}.$$

Let $\mu(\beta) = \alpha$. Then α is a primitive element in \mathbb{F}_q , and $\mu(\mathcal{T}) = \mathbb{F}_q$.

The Galois ring R has a cyclic Galois group of order m generated by the following Frobenius map σ :

$$\sigma(z) = \sigma(A + 2B) = A^2 + 2B^2.$$

The trace of z , $T(z)$, from R to \mathbb{Z}_4 is defined as

$$T(z) = \sum_{i=0}^{m-1} \sigma^i(z),$$

and $\text{tr}(x)$ is the ordinary trace function from \mathbb{F}_q to \mathbb{Z}_2 .

The Goethals code \mathcal{G} of length $q = 2^m$ over \mathbb{Z}_4 is a linear code with the following parity-check matrix

$$H_{\mathcal{G}} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & \beta & \beta^2 & \cdots & \beta^{2^m-2} \\ 0 & 2 & 2\beta^3 & 2\beta^6 & \cdots & 2\beta^{3(2^m-2)} \end{bmatrix}.$$

It is shown in [10, Theorem 25] that if m is odd, then the Goethals code \mathcal{G} has minimum Lee distance 8. The Delsarte-Goethals code \mathcal{DG} is defined as the dual of \mathcal{G} over \mathbb{Z}_4 , so its generator matrix is just $H_{\mathcal{G}}$.

Also the Delsarte-Goethals code has the following trace description. Let $\mathbf{c}(u, a, b)$, where $u \in \mathbb{Z}_4, a \in R, b \in \mathcal{T}$, be a vector in \mathbb{Z}_4^q indexed by the elements of \mathcal{T} such that $\mathbf{c}(u, a, b)_x = u + \text{T}(ax + 2bx^3)$ for all $x \in \mathcal{T}$, then

$$\mathcal{DG} = \{\mathbf{c}(u, a, b) \mid u \in \mathbb{Z}_4, a \in R, b \in \mathcal{T}\}.$$

The Lee weight of codeword $\mathbf{c}(u, a, b)$ can be expressed as

$$w_L(\mathbf{c}(u, a, b)) = q - \Re \left(i^u \sum_{x \in \mathcal{T}} i^{\text{T}(ax + 2bx^3)} \right). \quad (1)$$

Lemma 2.1 [13, Theorem 1] *Let $q = 2^m$ where m is odd. The Lee weight distribution of the Delsarte-Goethals code \mathcal{DG} is*

$$A_j = \begin{cases} 1, & \text{if } j = 0 \text{ or } 2q; \\ (q-1)q(2q-1)/6, & \text{if } j = q \pm \sqrt{2q}; \\ (q-1)2q(q+4)/2, & \text{if } j = q \pm \sqrt{q/2}; \\ (2q-1)(q^2-q+2), & \text{if } j = q. \end{cases}$$

The quaternary Kerdock code \mathcal{K} of length $q = 2^m$ is a subcode of \mathcal{DG} generated by

$$H_{\mathcal{K}} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & \beta & \beta^2 & \cdots & \beta^{2^m-2} \end{bmatrix}.$$

Hence

$$\mathcal{K} = \{\mathbf{c}(u, a, 0) \mid u \in \mathbb{Z}_4, a \in R\}.$$

The quaternary Preparata code \mathcal{P} is a code with parity-check matrix $H_{\mathcal{K}}$. If m is odd, it has minimum Lee weight 6.

Lemma 2.2 [13, Lemma 1] *Let $q = 2^m$ where $m \geq 3$ is odd. The Lee weight distribution of the Kerdock code \mathcal{K} is*

$$A_j = \begin{cases} 1, & \text{if } j = 0 \text{ or } 2q; \\ 2q(q-1), & \text{if } j = q \pm \sqrt{q/2}; \\ 4q-2, & \text{if } j = q. \end{cases}$$

The following relations on the Kerdock code \mathcal{K} will determine an abelian 4 class association scheme:

$$(x, y) \in \begin{cases} R_0, & \text{if } w_L(x - y) = 0; \\ R_1, & \text{if } w_L(x - y) = q - \sqrt{q/2}; \\ R_2, & \text{if } w_L(x - y) = q; \\ R_3, & \text{if } w_L(x - y) = q + \sqrt{q/2}; \\ R_4, & \text{if } w_L(x - y) = 2q. \end{cases} \quad (2)$$

Theorem 2.1 *The relations (2) on the codewords of quaternary Kerdock code define a 4 class abelian association scheme.*

The above scheme is just the dual of the scheme constructed in [6, Proposition 6].

3 Schemes related to the \mathcal{DG} code

It is natural to ask whether the obvious generalizations of the relations (2) on the Delsarte-Goethals code \mathcal{DG} will also give an abelian association scheme or not. It turns out that the answer is no. In order to get an association scheme, we should modify the relations a little:

$$(x, y) \in \begin{cases} S_0, & \text{if } w_L(x - y) = 0; \\ S_1, & \text{if } w_L(x - y) = q - \sqrt{2q}; \\ S_2, & \text{if } w_L(x - y) = q - \sqrt{q/2}; \\ S_3, & \text{if } w_L(x - y) = q \text{ and } x - y \in \mathcal{K}; \\ S_4, & \text{if } w_L(x - y) = q \text{ and } x - y \notin \mathcal{K}; \\ S_5, & \text{if } w_L(x - y) = q + \sqrt{q/2}; \\ S_6, & \text{if } w_L(x - y) = q + \sqrt{2q}; \\ S_7, & \text{if } w_L(x - y) = 2q. \end{cases} \quad (3)$$

Theorem 3.1 *The relations (3) on the codewords of quaternary Delsarte-Goethals code define a 7 class abelian association scheme \mathfrak{A} . The first and the second eigenmatrices are given in Appendix A.*

This is not the end of the story. Actually we can provide a more refined description of relations (3) to get an abelian 9 class association scheme as follows:

$$(x, y) \in \begin{cases} S_0, & \text{if } w_L(x - y) = 0; \\ S_1, & \text{if } w_L(x - y) = q - \sqrt{2q}; \\ S'_{21}, & \text{if } w_L(x - y) = q - \sqrt{q/2} \text{ and } x - y \in \mathcal{K}; \\ S'_{22}, & \text{if } w_L(x - y) = q - \sqrt{q/2} \text{ and } x - y \notin \mathcal{K}; \\ S_3, & \text{if } w_L(x - y) = q \text{ and } x - y \in \mathcal{K}; \\ S_4, & \text{if } w_L(x - y) = q \text{ and } x - y \notin \mathcal{K}; \\ S'_{51}, & \text{if } w_L(x - y) = q + \sqrt{q/2} \text{ and } x - y \notin \mathcal{K}; \\ S'_{52}, & \text{if } w_L(x - y) = q + \sqrt{q/2} \text{ and } x - y \in \mathcal{K}; \\ S_6, & \text{if } w_L(x - y) = q + \sqrt{2q}; \\ S_7, & \text{if } w_L(x - y) = 2q. \end{cases} \quad (4)$$

Theorem 3.2 *The relations (4) on the codewords of quaternary Delsarte-Goethals code define a 9 class abelian association scheme \mathfrak{B} . The first and the second eigenmatrices are given in Appendix A.*

We leave the proofs of the above two theorems to the end of this section, since they are immediate outcomes of Theorem 3.3. Recall the trace description of the Delsarte-Goethals code \mathcal{DG} , we know there is a one-to-one correspondence between the codewords of \mathcal{DG} and the set $\mathbb{Z}_4 \times R \times \mathcal{T}$ given by $(u, a, b) \longleftrightarrow \mathbf{c}(u, a, b)$. Since $\mu(\mathcal{T}) = \mathbb{F}_q$, there is a group isomorphism between the group $G = \mathbb{Z}_4 \times R \times \mathbb{F}_q$ and the \mathcal{DG} code. For $(u, a, b) \in G$, we introduce an exponential sum

$$S(u, a, b) = \sum_{X \in \mathcal{T}} i^{u + \text{Tr}(aX + 2bX^3)} + \sum_{X \in \mathcal{T}} i^{-u - \text{Tr}(aX + 2bX^3)},$$

here we have identified the element $b \in \mathbb{F}_q$ with its pre-image $\mu^{-1}(b) \in \mathcal{T}$. Now Eqn. (1) becomes

$$w_L(\mathbf{c}(u, a, b)) = q - S(u, a, b)/2. \quad (5)$$

So we can see that for $(u, a, b) \in G$,

$$S(u, a, b) \in \{\pm 2q, \pm 2\sqrt{2q}, \pm \sqrt{2q}, 0\},$$

from the weight distribution of Delsarte-Goethals code. According to the value of $S(u, a, b)$, we give a partition of G into ten parts as follows:

$$\begin{aligned}
\mathcal{R}_0 &= \{(u, a, b) \in G \mid S(u, a, b) = 2q\} = \{(0, 0, 0)\}, \\
\mathcal{R}_1 &= \{(u, a, b) \in G \mid S(u, a, b) = 2\sqrt{2q}\}, \\
\mathcal{R}_2 &= \{(u, a, b) \in G \mid S(u, a, b) = \sqrt{2q}, b = 0\}, \\
\mathcal{R}_3 &= \{(u, a, b) \in G \mid S(u, a, b) = \sqrt{2q}, b \neq 0\}, \\
\mathcal{R}_4 &= \{(u, a, b) \in G \mid S(u, a, b) = 0, b = 0\}, \\
\mathcal{R}_5 &= \{(u, a, b) \in G \mid S(u, a, b) = 0, b \neq 0\}, \\
\mathcal{R}_6 &= \{(u, a, b) \in G \mid S(u, a, b) = -\sqrt{2q}, b \neq 0\}, \\
\mathcal{R}_7 &= \{(u, a, b) \in G \mid S(u, a, b) = -\sqrt{2q}, b = 0\}, \\
\mathcal{R}_8 &= \{(u, a, b) \in G \mid S(u, a, b) = -2\sqrt{2q}\}, \text{ and} \\
\mathcal{R}_9 &= \{(u, a, b) \in G \mid S(u, a, b) = -2q\} = \{(2, 0, 0)\}.
\end{aligned}$$

Since the group G is abelian, its character group $\widehat{G} \cong G$. In order to describe the dual association scheme, we shall first provide the dual partition of the group $\widehat{G} = G$. For convenience, we will use capital letters such as X, Y to denote elements in \mathcal{T} , and the corresponding lower cases to represent their respective projections modulo 2 in \mathbb{F}_q . The dual partition is as follows:

$$\begin{aligned}
\mathcal{E}_0 &= \{(0, 0, 0)\}, \\
\mathcal{E}_1 &= \{(0, 0, r) \mid r \in \mathbb{F}_q^*\}, \\
\mathcal{E}_2 &= \{(1, X, x^3) \mid X \in \mathcal{T}\} \cup \{(3, -X, x^3) \mid X \in \mathcal{T}\}, \\
\mathcal{E}_3 &= (\{(1, X, r) \mid X \in \mathcal{T}, r \in \mathbb{F}_q\} \cup \{(3, -X, r) \mid X \in \mathcal{T}, r \in \mathbb{F}_q\}) \setminus \mathcal{E}_2, \\
\mathcal{E}_4 &= \{(0, -X + Y, x^3 + y^3) \mid X, Y \in \mathcal{T}, X \neq Y\} \\
&\quad \cup \{(2, X + Y, x^3 + y^3) \mid X, Y \in \mathcal{T}\} \\
&\quad \cup \{(2, -X - Y, x^3 + y^3) \mid X, Y \in \mathcal{T}\}, \\
\mathcal{E}_5 &= (\{(0, S, r) \mid S \in R \setminus 2R, r \in \mathbb{F}_q\} \cup \{(2, S, r) \mid S \in R, r \in \mathbb{F}_q\}) \setminus \mathcal{E}_4, \\
\mathcal{E}_6 &= \{(1, 2X - Y, y^3) \mid X, Y \in \mathcal{T}, X \neq Y\} \\
&\quad \cup \{(1, -X - Y - Z, x^3 + y^3 + z^3) \mid X, Y, Z \in \mathcal{T} \text{ are pairwise distinct}\} \\
&\quad \cup \{(1, X + Y - Z, x^3 + y^3 + z^3) \mid X, Y, Z \in \mathcal{T} \text{ are pairwise distinct}\} \\
&\quad \cup \{(3, 2X + Y, y^3) \mid X, Y \in \mathcal{T}, X \neq Y\} \\
&\quad \cup \{(3, X + Y + Z, x^3 + y^3 + z^3) \mid X, Y, Z \in \mathcal{T} \text{ are pairwise distinct}\} \\
&\quad \cup \{(3, -X - Y + Z, x^3 + y^3 + z^3) \mid X, Y, Z \in \mathcal{T} \text{ are pairwise distinct}\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_7 &= \{(1, 2X - Y, r) \mid X, Y \in \mathcal{T}, X \neq Y, r \neq y^3\} \\
&\cup \{(1, -X - Y - Z, r) \mid X, Y, Z \in \mathcal{T} \text{ are pairwise distinct}, r \neq x^3 + y^3 + z^3\} \\
&\cup \{(1, X + Y - Z, r) \mid X, Y, Z \in \mathcal{T} \text{ are pairwise distinct}, r \neq x^3 + y^3 + z^3\} \\
&\cup \{(3, 2X + Y, r) \mid X, Y \in \mathcal{T}, X \neq Y, r \neq y^3\} \\
&\cup \{(3, X + Y + Z, r) \mid X, Y, Z \in \mathcal{T} \text{ are pairwise distinct}, r \neq x^3 + y^3 + z^3\} \\
&\cup \{(3, -X - Y + Z, r) \mid X, Y, Z \in \mathcal{T} \text{ are pairwise distinct}, r \neq x^3 + y^3 + z^3\}, \\
\mathcal{E}_8 &= \{(0, 2X, \sum_{i=1}^4 y_i^3) \mid X \in \mathcal{T}^*, Y_i \in \mathcal{T}, 2X = \sum_{i=1}^4 Y_i \text{ or } 2X = Y_1 + Y_2 - Y_3 - Y_4\}, \text{ and} \\
\mathcal{E}_9 &= \{(0, 2X, r) \mid X \in \mathcal{T}^*, r \in \mathbb{F}_q\} \setminus \mathcal{E}_8.
\end{aligned}$$

Remark: Let m be an odd integer. In the study of dimensional dual hyperovals, Pasini and Yoshiara [19, Proposition 1.7] showed that the Cayley graph of the following set

$$S = \{(1, x, x^3) \mid x \in \mathbb{F}_q\} \subset \mathbb{Z}_2 \times \mathbb{F}_q \times \mathbb{F}_q$$

is a distance regular graph of diameter 4. This is analogous to our set \mathcal{E}_2 above.

Theorem 3.3 *Let $G = \mathbb{Z}_4 \times R \times \mathbb{F}_q$, and define the binary relations $R_i = \{(g, h) \mid g - h \in \mathcal{R}_i\}$ for $i = 0, \dots, 9$. Then $\mathfrak{B}' = (G; R_i, 0 \leq i \leq 9)$ is a 9-class association scheme, with the first and the second eigenmatrices given by P and Q (listed in Appendix A). The binary relations $R'_i = \{(g, h) \mid g - h \in \mathcal{E}_i\}$ for $i = 0, \dots, 9$ define an association scheme which is dual to \mathfrak{B}' , so it has the first and second eigenmatrices: $P' = Q, Q' = P$.*

Proof. Denote $s = \sqrt{2q}$. The element of the group ring $\mathbb{C}G$ will be written as $\sum_{(u,a,b) \in G} c(u, a, b)[(u, a, b)]$, where $c(u, a, b) \in \mathbb{C}$. We set

$$\begin{aligned}
\mathcal{N}_{2i} &= \sum_{(u,a,b) \in G} S(u, a, b)^i [(u, a, b)], \text{ and} \\
\mathcal{N}_{2i+1} &= \sum_{\substack{(u,a,b) \in G \\ b=0}} S(u, a, b)^i [(u, a, b)],
\end{aligned}$$

for $i = 0, 1, \dots, 4$. The above transformation can be written in the matrix form as

$$(\mathcal{N}_0, \mathcal{N}_1, \dots, \mathcal{N}_9) = (\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_9) T,$$

where

$$T = \begin{pmatrix} 1 & 1 & s^2 & s^2 & s^4 & s^4 & s^6 & s^6 & s^8 & s^8 \\ 1 & 0 & 2s & 0 & 4s^2 & 0 & 8s^3 & 0 & 16s^4 & 0 \\ 1 & 1 & s & s & s^2 & s^2 & s^3 & s^3 & s^4 & s^4 \\ 1 & 0 & s & 0 & s^2 & 0 & s^3 & 0 & s^4 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -s & 0 & s^2 & 0 & -s^3 & 0 & s^4 & 0 \\ 1 & 1 & -s & -s & s^2 & s^2 & -s^3 & -s^3 & s^4 & s^4 \\ 1 & 0 & -2s & 0 & 4s^2 & 0 & -8s^3 & 0 & 16s^4 & 0 \\ 1 & 1 & -s^2 & -s^2 & s^4 & s^4 & -s^6 & -s^6 & s^8 & s^8 \end{pmatrix}.$$

Using Maple, it is easy to compute that $\det(T) = -1152 s^{23} (s-1)^2 (s+1)^2$, so T is invertible. The key step in our proof is the completion of the following character table \mathfrak{T} :

$$\begin{array}{c} \mathcal{E}_0 \\ \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \\ \mathcal{E}_4 \\ \mathcal{E}_5 \\ \mathcal{E}_6 \\ \mathcal{E}_7 \\ \mathcal{E}_8 \\ \mathcal{E}_9 \end{array} \begin{pmatrix} \mathcal{N}_0 & \mathcal{N}_1 & \mathcal{N}_2 & \mathcal{N}_3 & \mathcal{N}_4 & \mathcal{N}_5 & \mathcal{N}_6 & \mathcal{N}_7 & \mathcal{N}_8 & \mathcal{N}_9 \\ 4q^3 & 4q^2 & 0 & 0 & 8q^4 & 8q^3 & 0 & 0 & 16q^4(3q-1) & 16q^3(3q-1) \\ 0 & 4q^2 & 0 & 0 & 0 & 8q^3 & 0 & 0 & 0 & 16q^3(3q-1) \\ 0 & 0 & 4q^3 & 4q^2 & 0 & 0 & 8q^3(3q-1) & 8q^2(3q-1) & 0 & 0 \\ 0 & 0 & 0 & 4q^2 & 0 & 0 & 0 & 8q^2(3q-1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 8q^3 & 8q^2 & 0 & 0 & 32q^3(3q-2) & 32q^4 \\ 0 & 0 & 0 & 0 & 0 & 8q^2 & 0 & 0 & 32q^3(q-2) & 32q^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 24q^3 & 8q^2(2q-1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8q^2(2q-1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48q^4 & 16q^3(2q-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16q^3(2q-1) \end{pmatrix}.$$

The next section is devoted to the computation of this table. Taking the column indexed by \mathcal{N}_6 as an illustration, we can see that

$$g(\mathcal{N}_6) = \begin{cases} 8q^3(3q-1), & \text{if } g \in \mathcal{E}_2; \\ 24q^3, & \text{if } g \in \mathcal{E}_6; \\ 0, & \text{otherwise.} \end{cases}$$

Thus the character table P is obtained by multiplying the above character table \mathfrak{T} with the matrix T^{-1} from the left. Now the assertion that $\mathfrak{B}' = (G; R_i, 0 \leq i \leq 9)$ is a 9-class association scheme is an immediate consequence of the Bannai-Muzychuk criterion [3, 18]. \square

Proof of Theorems 3.1-3.2. From the statements at the beginning of this section, Theorem 3.2 is clear. Theorem 3.1 follows directly from the Bannai-Muzychuk criterion and the eigenmatrices of association scheme \mathfrak{B} . The scheme \mathfrak{A} is a fusion scheme of \mathfrak{B} . \square

Corollary 3.1 *There exists a 5-class association scheme \mathcal{C} on the quotient group $G/\langle(2,0,0)\rangle$. Furthermore, there exists a 4-class fusion scheme \mathcal{D} of the scheme \mathcal{C} . Their first and second eigenmatrices are given in Appendix A.*

Proof. This can be readily checked using the eigenmatrices of the association scheme \mathcal{B}' with the help of Bannai-Muzychuk criterion. \square

4 Completion of the character table \mathfrak{T}

4.1 Columns indexed by $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2$ and \mathcal{N}_3

The first four columns of the character table \mathfrak{T} can be obtained through direct computations. Here we only provide the proof of the column indexed by \mathcal{N}_2 , each of the other three cases is exactly the same.

For $g = (v, c, d) \in G$, we have

$$\begin{aligned}
g(\mathcal{N}_2) &= \sum_{(u,a,b) \in G} S(u, a, b) i^{uv+T(ac+2bd)} \\
&= \sum_{(u,a,b) \in G} \sum_{X \in \mathcal{T}} i^{u(v+1)+T(a(c+X))+2T(b(d+X^3))} + \\
&\quad \sum_{(u,a,b) \in G} \sum_{X \in \mathcal{T}} i^{u(v-1)+T(a(c-X))+2T(b(d-X^3))} \\
&= \sum_{X \in \mathcal{T}} \sum_{u \in \mathbb{Z}_4} i^{u(v+1)} \sum_{a \in R} i^{T(a(c+X))} \sum_{b \in \mathcal{T}} i^{2T(b(d+X^3))} + \\
&\quad \sum_{X \in \mathcal{T}} \sum_{u \in \mathbb{Z}_4} i^{u(v-1)} \sum_{a \in R} i^{T(a(c-X))} \sum_{b \in \mathcal{T}} i^{2T(b(d-X^3))} \\
&= \begin{cases} 4q^3, & \text{if } g \in \mathcal{E}_2; \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

and this completes our proof.

4.2 Columns indexed by $\mathcal{N}_4, \mathcal{N}_5, \mathcal{N}_6, \mathcal{N}_7$ and \mathcal{N}_9

First, a direct calculation gives that

$$\begin{aligned}
g(\mathcal{N}_4) &= \sum_{(u,a,b) \in G} S(u, a, b)^2 i^{uv+T(ac+2bd)} \\
&= \sum_{u \in \mathbb{Z}_4} i^{u(v+2)} \sum_{X,Y \in \mathcal{T}} \sum_{a \in R} i^{T(a(c+X+Y))} \sum_{b \in \mathbb{F}_q} i^{T(2b(d+X^3+Y^3))} + \\
&\quad \sum_{u \in \mathbb{Z}_4} i^{u(v-2)} \sum_{X,Y \in \mathcal{T}} \sum_{a \in R} i^{T(a(c-X-Y))} \sum_{b \in \mathbb{F}_q} i^{T(2b(d-X^3-Y^3))} + \\
&\quad 2 \sum_{u \in \mathbb{Z}_4} i^{uv} \sum_{X,Y \in \mathcal{T}} \sum_{a \in R} i^{T(a(c+X-Y))} \sum_{b \in \mathbb{F}_q} i^{T(2b(d+X^3-Y^3))}
\end{aligned}$$

for $g = (v, c, d) \in G$.

Suppose that $v = 0$ and $c = -Z + W$ for some $Z, W \in \mathcal{T}$ with $Z \neq W$. Then only the last term in the above sum possibly contributes. Using (b) of Lemma B.2, we see that it will contribute $8q^3$ if $d = z^3 + w^3$ and zero otherwise. A routine analysis shows that

$$g(\mathcal{N}_4) = \begin{cases} 8q^4, & \text{if } g \in \mathcal{E}_0; \\ 8q^3, & \text{if } g \in \mathcal{E}_4; \\ 0, & \text{otherwise,} \end{cases}$$

and this completes the column indexed by \mathcal{N}_4 .

The column indexed by \mathcal{N}_5 can be checked exactly as same as \mathcal{N}_4 .

Now we continue to treat the column \mathcal{N}_6 :

$$\begin{aligned}
g(\mathcal{N}_6) &= \sum_{(u,a,b) \in G} S(u, a, b)^3 i^{uv+T(ac+2bd)} \\
&= \sum_{u \in \mathbb{Z}_4} i^{u(v+3)} \sum_{X,Y,Z \in \mathcal{T}} \sum_{a \in R} i^{T(a(c+X+Y+Z))} \sum_{b \in \mathbb{F}_q} i^{T(2b(d+X^3+Y^3+Z^3))} + \\
&\quad \sum_{u \in \mathbb{Z}_4} i^{u(v-3)} \sum_{X,Y,Z \in \mathcal{T}} \sum_{a \in R} i^{T(a(c-X-Y-Z))} \sum_{b \in \mathbb{F}_q} i^{T(2b(d-X^3-Y^3-Z^3))} + \\
&\quad 3 \sum_{u \in \mathbb{Z}_4} i^{u(v+1)} \sum_{X,Y,Z \in \mathcal{T}} \sum_{a \in R} i^{T(a(c+X+Y-Z))} \sum_{b \in \mathbb{F}_q} i^{T(2b(d+X^3+Y^3-Z^3))} + \\
&\quad 3 \sum_{u \in \mathbb{Z}_4} i^{u(v-1)} \sum_{X,Y,Z \in \mathcal{T}} \sum_{a \in R} i^{T(a(c-X-Y+Z))} \sum_{b \in \mathbb{F}_q} i^{T(2b(d-X^3-Y^3+Z^3))}.
\end{aligned}$$

First, suppose that $g = (1, W, w^3) \in \mathcal{E}_2$ for some $W \in \mathcal{T}$. Then only the first and last terms in the above sum will contribute. Using (d) of Lemma B.2, we see that the first term will contribute $4q^3$ if $d = w^3$ and zero otherwise. Similar analysis shows that the last term will contribute $12q^3(2q-1)$ if $d = w^3$ and zero otherwise by (c) of Lemma B.2. The analysis for $g = (3, -W, w^3) \in \mathcal{E}_2$ is exactly the same. So $g(\mathcal{N}_6) = 8q^3(3q-1)$ for $g \in \mathcal{E}_2$. Secondly, when $g \in \mathcal{E}_6$, the argument can be proved similarly using Lemmas B.6-B.7 in Appendix B.

The column indexed by \mathcal{N}_7 and \mathcal{N}_9 are readily completed by using Lemma B.3 and Corollary B.1 respectively.

4.3 Column indexed by \mathcal{N}_8

This is the most difficult case. When $g = (0, 0, 0)$, the identity $g(\mathcal{N}_8) = 16q^4(3q-1)$ is direct. When $g \in \mathcal{E}_8$, we can prove $g(\mathcal{N}_8) = 48q^4$ by Corollaries B.2-B.3. The exponential sum $\xi(a, b) = \sum_{X \in \mathcal{T}} i^{\text{T}(aX + 2bX^3)}$ is closely related to $S(u, a, b)$. We further introduce two exponential sums

$$\mathbf{E}(c, d) := \sum_{a \in R} \sum_{b \in \mathbb{F}_q} \left(\xi^4(a, b) + \overline{\xi^4(a, b)} + 6\xi^2(a, b)\overline{\xi^2(a, b)} \right) i^{\text{T}(ac + 2bd)}$$

and

$$\mathbf{F}(c, d) := \sum_{a \in R} \sum_{b \in \mathbb{F}_q} \left(\xi^3(a, b)\overline{\xi(a, b)} + \xi(a, b)\overline{\xi^3(a, b)} \right) i^{uv + \text{T}(ac + 2bd)}.$$

Then

$$g(\mathcal{N}_8) = \sum_{(u, a, b) \in G} S(u, a, b)^4 i^{uv + \text{T}(ac + 2bd)} = \sum_{u \in \mathbb{Z}_4} i^{uv} \mathbf{E}(c, d) + 4 \sum_{u \in \mathbb{Z}_4} i^{u(v+2)} \mathbf{F}(c, d).$$

So it is enough to determine the distribution of $\mathbf{E}(c, d)$ and $\mathbf{F}(c, d)$. Since the determination is very technical and complex, we prefer to leave it in Appendix B. Using Lemmas C.2-C.4, it is now a routine check to see that

$$g(\mathcal{N}_8) = \begin{cases} 2^{3m+6}(3 \cdot 2^{m-1} - 1), & \text{if } g \in \mathcal{E}_4; \\ 2^{3m+6}(2^{m-1} - 1), & \text{if } g \in \mathcal{E}_5. \end{cases}$$

5 Conclusion

In this paper, we construct a 9-class scheme from a refinement of the partition of the \mathcal{DG} code by its Lee weights. We get the explicit expressions of the dual scheme of this 9-class scheme by manipulations of complicated exponential sums and heavy computations. There is an interesting “non-symmetry” between the

characterization of the 9-class scheme and its dual in the sense that the description of the original scheme reflects the properties of the underlying code while we see nothing about the code in the description of the dual scheme. Moreover, the dual scheme can not be described by the cosets of the Goethals codes as far as we see it. It will be interesting to see what code properties are reflected in the dual scheme.

In [6] Bonnecaze and Duursma showed that the partition of the Kerdock (resp. shortened Kerdock) code induced by the complete weight enumerators gives rise to an association scheme. Using this scheme and the complete weight enumerator of the Kerdock (resp. shortened Kerdock) code, they also showed that the complete weight enumerator of each coset of the dual code, namely Preparata (resp. punctured Preparata) code can be explicitly determined. Thus it is reasonable to believe that this is also true for the \mathcal{DG} code, which would give rise to a 22-class association scheme. Since the complete weight enumerator of the \mathcal{DG} code has been explicitly determined by Shin, Kumar and Helleseeth ([20]), once we figure out the parameters of this conjectured scheme, then theoretically we know all about the complete weight enumerators of each coset of the Goethals code. The coset weight enumerators of the Goethals code have been studied by Helleseeth and Zinoviev (see [14, 15]). We mention that our 9-class scheme is not a fusion scheme of this conjectured 22-class scheme.

Davis and Xiang (see [8]) constructed the first known examples where the non-homomorphic bijection approach outlined by Hagita and Schmidt (see [9]) can produce negative Latin square type partial difference sets in groups that previously had no known constructions. The Cayley graphs of partial difference sets are strongly regular, so yield two-class association schemes. Therefore it is interesting to investigate whether there are translation schemes over elementary abelian 2-groups with the same parameters as those constructed in [1, 6] and this paper from the various \mathbb{Z}_4 -linear codes. When $m = 3$, the Gray map image of the \mathcal{DG} code is a \mathbb{Z}_2 -linear code, and we checked that the Gray map image of the 9-class scheme remains a scheme but with different parameters. However, when $m > 3$, the binary \mathcal{DG} code is no longer linear, and no scheme arises naturally in this way.

6 Appendix A

The first and second eigenmatrices of association scheme \mathfrak{A} :

$$\begin{aligned}
 P = & \begin{array}{c} \mathcal{E}_0 \\ \mathcal{E}_1 \cup \mathcal{E}_9 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \cup \mathcal{E}_7 \\ \mathcal{E}_4 \\ \mathcal{E}_5 \\ \mathcal{E}_6 \\ \mathcal{E}_8 \end{array} \begin{pmatrix} \mathcal{R}_0 & \mathcal{R}_1 & \mathcal{R}_2 \cup \mathcal{R}_3 & \mathcal{R}_4 \\ 1 & 1/24 s^6 - 1/8 s^4 + 1/12 s^2 & 1/2 s^4 - 4/3 s^2 + 1/12 s^6 & -2 + 2 s^2 \\ 1 & -1/12 s^4 + 1/12 s^2 & 1/3 s^4 - 4/3 s^2 & -2 + 2 s^2 \\ 1 & 1/12 s^5 - 1/4 s^3 + 1/6 s & 1/2 s^3 - 4/3 s + 1/12 s^5 & 0 \\ 1 & -1/6 s^3 + 1/6 s & 1/3 s^3 - 4/3 s & 0 \\ 1 & 1/8 s^4 - 1/4 s^2 & 0 & -2 \\ 1 & -1/4 s^2 & 0 & -2 \\ 1 & 1/12 s^3 + 1/6 s & -4/3 s - 1/6 s^3 & 0 \\ 1 & 1/24 s^4 + 1/12 s^2 & -4/3 s^2 - 1/6 s^4 & -2 + 2 s^2 \end{pmatrix} \\
 & \begin{array}{c} \mathcal{R}_5 \\ \mathcal{R}_6 \cup \mathcal{R}_7 \\ \mathcal{R}_8 \\ \mathcal{R}_9 \end{array} \begin{pmatrix} 1/4 s^6 - 3/4 s^4 + 1/2 s^2 & 1/2 s^4 - 4/3 s^2 + 1/12 s^6 & 1/24 s^6 - 1/8 s^4 + 1/12 s^2 & 1 \\ -1/2 s^4 + 1/2 s^2 & 1/3 s^4 - 4/3 s^2 & -1/12 s^4 + 1/12 s^2 & 1 \\ 0 & -1/12 s^5 + 4/3 s - 1/2 s^3 & -1/12 s^5 + 1/4 s^3 - 1/6 s & -1 \\ 0 & -1/3 s^3 + 4/3 s & 1/6 s^3 - 1/6 s & -1 \\ -1/4 s^4 + 1/2 s^2 & 0 & 1/8 s^4 - 1/4 s^2 & 1 \\ 1/2 s^2 & 0 & -1/4 s^2 & 1 \\ 0 & 1/6 s^3 + 4/3 s & -1/12 s^3 - 1/6 s & -1 \\ 1/4 s^4 + 1/2 s^2 & -4/3 s^2 - 1/6 s^4 & 1/24 s^4 + 1/12 s^2 & 1 \end{pmatrix}, \\
 Q = & \begin{pmatrix} 1 & 1/2 s^2 - 1 & s^2 & 1/2 (s^2 - 2) s^2 & 1/2 s^2 (s^2 - 1) \\ 1 & -1 & 2 s & -2 s & 3/2 s^2 \\ 1 & 1/2 s^2 - 1 & s & 1/2 (s^2 - 2) s & 0 \\ 1 & -1 & s & -s & 0 \\ 1 & 1/2 s^2 - 1 & 0 & 0 & -1/2 s^2 \\ 1 & -1 & 0 & 0 & -1/2 s^2 \\ 1 & -1 & -s & s & 0 \\ 1 & 1/2 s^2 - 1 & -s & -1/2 (s^2 - 2) s & 0 \end{pmatrix} \\
 & \begin{pmatrix} 1/4 s^2 (s^4 - 3 s^2 + 2) & 1/6 s^2 (s^4 - 3 s^2 + 2) & 1/6 s^4 - 1/2 s^2 + 1/3 \\ -3/2 s^2 & 1/3 s (s^2 + 2) & 1/6 s^2 + 1/3 \\ 0 & -1/3 s (s^2 - 1) & 1/3 - 1/3 s^2 \\ -1/4 (s^2 - 2) s^2 & 0 & 1/6 s^4 - 1/2 s^2 + 1/3 \\ 1/2 s^2 & 0 & 1/6 s^2 + 1/3 \\ 0 & 1/3 s (s^2 - 1) & 1/3 - 1/3 s^2 \\ -3/2 s^2 & -1/3 s (s^2 + 2) & 1/6 s^2 + 1/3 \\ 1/4 s^2 (s^4 - 3 s^2 + 2) & -1/6 s^2 (s^4 - 3 s^2 + 2) & 1/6 s^4 - 1/2 s^2 + 1/3 \end{pmatrix}.
 \end{aligned}$$

The first and the second eigenmatrices of association schemes \mathfrak{B} and \mathfrak{B}' :

$$\begin{aligned}
P = & \begin{array}{c} \varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \\ \varepsilon_7 \\ \varepsilon_8 \\ \varepsilon_9 \end{array} \begin{pmatrix} \mathcal{R}_0 & \mathcal{R}_1 & \mathcal{R}_2 & \mathcal{R}_3 & \mathcal{R}_4 \\ 1 & 1/24 s^6 - 1/8 s^4 + 1/12 s^2 & 1/2 s^4 - s^2 & 1/12 s^6 - 1/3 s^2 & -2 + 2 s^2 \\ 1 & -1/12 s^4 + 1/12 s^2 & 1/2 s^4 - s^2 & -1/6 s^4 - 1/3 s^2 & -2 + 2 s^2 \\ 1 & 1/12 s^5 - 1/4 s^3 + 1/6 s & 1/2 s^3 - s & 1/12 s^5 - 1/3 s & 0 \\ 1 & -1/6 s^3 + 1/6 s & 1/2 s^3 - s & -1/6 s^3 - 1/3 s & 0 \\ 1 & 1/8 s^4 - 1/4 s^2 & 0 & 0 & -2 \\ 1 & -1/4 s^2 & 0 & 0 & -2 \\ 1 & 1/12 s^3 + 1/6 s & -s & -1/6 s^3 - 1/3 s & 0 \\ 1 & -1/6 s^3 + 1/6 s & -s & -1/3 s + 1/3 s^3 & 0 \\ 1 & 1/24 s^4 + 1/12 s^2 & -s^2 & -1/6 s^4 - 1/3 s^2 & -2 + 2 s^2 \\ 1 & -1/12 s^4 + 1/12 s^2 & -s^2 & -1/3 s^2 + 1/3 s^4 & -2 + 2 s^2 \end{pmatrix} \\
& \begin{array}{c} \mathcal{R}_5 \\ \mathcal{R}_6 \\ \mathcal{R}_7 \\ \mathcal{R}_8 \\ \mathcal{R}_9 \end{array} \begin{pmatrix} 1/4 s^6 - 3/4 s^4 + 1/2 s^2 & 1/12 s^6 - 1/3 s^2 & 1/2 s^4 - s^2 & 1/24 s^6 - 1/8 s^4 + 1/12 s^2 & 1 \\ -1/2 s^4 + 1/2 s^2 & -1/6 s^4 - 1/3 s^2 & 1/2 s^4 - s^2 & -1/12 s^4 + 1/12 s^2 & 1 \\ 0 & -1/12 s^5 + 1/3 s & -1/2 s^3 + s & -1/12 s^5 + 1/4 s^3 - 1/6 s & -1 \\ 0 & 1/6 s^3 + 1/3 s & -1/2 s^3 + s & 1/6 s^3 - 1/6 s & -1 \\ -1/4 s^4 + 1/2 s^2 & 0 & 0 & 1/8 s^4 - 1/4 s^2 & 1 \\ 1/2 s^2 & 0 & 0 & -1/4 s^2 & 1 \\ 0 & 1/6 s^3 + 1/3 s & s & -1/12 s^3 - 1/6 s & -1 \\ 0 & 1/3 s - 1/3 s^3 & s & 1/6 s^3 - 1/6 s & -1 \\ 1/4 s^4 + 1/2 s^2 & -1/6 s^4 - 1/3 s^2 & -s^2 & 1/24 s^4 + 1/12 s^2 & 1 \\ -1/2 s^4 + 1/2 s^2 & -1/3 s^2 + 1/3 s^4 & -s^2 & -1/12 s^4 + 1/12 s^2 & 1 \end{pmatrix}, \\
Q = & \begin{pmatrix} 1 & 1/2 s^2 - 1 & s^2 & 1/2 (s^2 - 2) s^2 & 1/2 s^2 (s^2 - 1) & 1/4 s^2 (s^4 - 3 s^2 + 2) \\ 1 & -1 & 2 s & -2 s & 3/2 s^2 & -3/2 s^2 \\ 1 & 1/2 s^2 - 1 & s & 1/2 (s^2 - 2) s & 0 & 0 \\ 1 & -1 & s & -s & 0 & 0 \\ 1 & 1/2 s^2 - 1 & 0 & 0 & -1/2 s^2 & -1/4 (s^2 - 2) s^2 \\ 1 & -1 & 0 & 0 & -1/2 s^2 & 1/2 s^2 \\ 1 & -1 & -s & s & 0 & 0 \\ 1 & 1/2 s^2 - 1 & -s & -1/2 (s^2 - 2) s & 0 & 0 \\ 1 & -1 & -2 s & 2 s & 3/2 s^2 & -3/2 s^2 \\ 1 & 1/2 s^2 - 1 & -s^2 & -1/2 (s^2 - 2) s^2 & 1/2 s^2 (s^2 - 1) & 1/4 s^2 (s^4 - 3 s^2 + 2) \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix}
1/6 s^2 (s^4 - 3 s^2 + 2) & 1/12 s^2 (s^4 - 4) & 1/6 s^4 - 1/2 s^2 + 1/3 & 1/12 s^4 - 1/3 \\
1/3 s (s^2 + 2) & -1/3 s (s^2 + 2) & 1/6 s^2 + 1/3 & -1/6 s^2 - 1/3 \\
-1/3 s (s^2 - 1) & -1/6 s (s^2 + 2) & 1/3 - 1/3 s^2 & -1/6 s^2 - 1/3 \\
-1/3 s (s^2 - 1) & 1/3 s (s^2 - 1) & 1/3 - 1/3 s^2 & -1/3 + 1/3 s^2 \\
0 & 0 & 1/6 s^4 - 1/2 s^2 + 1/3 & 1/12 s^4 - 1/3 \\
0 & 0 & 1/6 s^2 + 1/3 & -1/6 s^2 - 1/3 \\
1/3 s (s^2 - 1) & -1/3 s (s^2 - 1) & 1/3 - 1/3 s^2 & -1/3 + 1/3 s^2 \\
1/3 s (s^2 - 1) & 1/6 s (s^2 + 2) & 1/3 - 1/3 s^2 & -1/6 s^2 - 1/3 \\
-1/3 s (s^2 + 2) & 1/3 s (s^2 + 2) & 1/6 s^2 + 1/3 & -1/6 s^2 - 1/3 \\
-1/6 s^2 (s^4 - 3 s^2 + 2) & -1/12 s^2 (s^4 - 4) & 1/6 s^4 - 1/2 s^2 + 1/3 & 1/12 s^4 - 1/3
\end{pmatrix}.$$

The first and the second eigenmatrices of association scheme \mathcal{C} :

$$P = \begin{pmatrix}
1 & 1/24 s^6 - 1/8 s^4 + 1/12 s^2 & 1/2 s^4 - s^2 & 1/12 s^6 - 1/3 s^2 & s^2 - 1 & 1/8 s^6 - 3/8 s^4 + 1/4 s^2 \\
1 & -1/12 s^4 + 1/12 s^2 & 1/2 s^4 - s^2 & -1/6 s^4 - 1/3 s^2 & s^2 - 1 & -1/4 s^4 + 1/4 s^2 \\
1 & 1/8 s^4 - 1/4 s^2 & 0 & 0 & -1 & -1/8 s^4 + 1/4 s^2 \\
1 & -1/4 s^2 & 0 & 0 & -1 & 1/4 s^2 \\
1 & 1/24 s^4 + 1/12 s^2 & -s^2 & -1/6 s^4 - 1/3 s^2 & s^2 - 1 & 1/8 s^4 + 1/4 s^2 \\
1 & -1/12 s^4 + 1/12 s^2 & -s^2 & -1/3 s^2 + 1/3 s^4 & s^2 - 1 & -1/4 s^4 + 1/4 s^2
\end{pmatrix},$$

$$Q = \begin{pmatrix}
1 & 1/2 s^2 - 1 & 1/2 s^2 (s^2 - 1) & 1/4 s^2 (s^4 - 3 s^2 + 2) & 1/6 s^4 - 1/2 s^2 + 1/3 & 1/12 s^4 - 1/3 \\
1 & -1 & 3/2 s^2 & -3/2 s^2 & 1/6 s^2 + 1/3 & -1/6 s^2 - 1/3 \\
1 & 1/2 s^2 - 1 & 0 & 0 & 1/3 - 1/3 s^2 & -1/6 s^2 - 1/3 \\
1 & -1 & 0 & 0 & 1/3 - 1/3 s^2 & -1/3 + 1/3 s^2 \\
1 & 1/2 s^2 - 1 & -1/2 s^2 & -1/4 (s^2 - 2) s^2 & 1/6 s^4 - 1/2 s^2 + 1/3 & 1/12 s^4 - 1/3 \\
1 & -1 & -1/2 s^2 & 1/2 s^2 & 1/6 s^2 + 1/3 & -1/6 s^2 - 1/3
\end{pmatrix}.$$

The first and the second eigenmatrices of association scheme \mathcal{D} :

$$P = \begin{pmatrix}
1 & 1/24 s^6 - 1/8 s^4 + 1/12 s^2 & 1/2 s^4 - 4/3 s^2 + 1/12 s^6 & s^2 - 1 & 1/8 s^6 - 3/8 s^4 + 1/4 s^2 \\
1 & -1/12 s^4 + 1/12 s^2 & 1/3 s^4 - 4/3 s^2 & s^2 - 1 & -1/4 s^4 + 1/4 s^2 \\
1 & 1/8 s^4 - 1/4 s^2 & 0 & -1 & -1/8 s^4 + 1/4 s^2 \\
1 & -1/4 s^2 & 0 & -1 & 1/4 s^2 \\
1 & 1/24 s^4 + 1/12 s^2 & -4/3 s^2 - 1/6 s^4 & s^2 - 1 & 1/8 s^4 + 1/4 s^2
\end{pmatrix},$$

$$Q = \begin{pmatrix}
1 & 1/2 s^2 - 4/3 + 1/12 s^4 & 1/2 s^2 (s^2 - 1) & 1/4 s^2 (s^4 - 3 s^2 + 2) & 1/6 s^4 - 1/2 s^2 + 1/3 \\
1 & -4/3 - 1/6 s^2 & 3/2 s^2 & -3/2 s^2 & 1/6 s^2 + 1/3 \\
1 & 1/3 s^2 - 4/3 & 0 & 0 & 1/3 - 1/3 s^2 \\
1 & 1/2 s^2 - 4/3 + 1/12 s^4 & -1/2 s^2 & -1/4 (s^2 - 2) s^2 & 1/6 s^4 - 1/2 s^2 + 1/3 \\
1 & -4/3 - 1/6 s^2 & -1/2 s^2 & 1/2 s^2 & 1/6 s^2 + 1/3
\end{pmatrix}.$$

7 Appendix B

Lemma B.1 [12, Lemma 2] Let $\mathbf{e} = (e_X)_{X \in \mathcal{T}}$ and let $E_j = \{X \mid e_X = j\}$ for $j = 0, 1, 2, 3$. The equation given by

$$\sum_{X \in \mathcal{T}} e_X X = A + 2B, \quad A, B \in \mathcal{T}, e_X \in \mathbb{Z}_4$$

is equivalent to the two binary equations

$$a = \sum_{X \in E_1 \cup E_3} x$$

and

$$b^2 = \sum_{X \in E_2 \cup E_3} x^2 + \sum_{\substack{X, Y \in E_1 \cup E_3 \\ X < Y}} xy.$$

We set $2R = \{2x \mid x \in R\}$ and $-\mathcal{T} = \{-X \mid X \in \mathcal{T}\}$.

Lemma B.2 [6, Theorem 1] Let $R = GR(4, m)$, $m > 0$, and \mathcal{T} be the Teichmüller set.

- (a) The multiset $\mathcal{T} + 2\mathcal{T} = \{X + 2Y \mid X, Y \in \mathcal{T}\}$ contains each element of R with multiplicity one.
- (b) The multiset $\mathcal{T} - \mathcal{T} = \{X - Y \mid X, Y \in \mathcal{T}\}$ contains 0 with multiplicity 2^m , no other elements of $2R$, and the elements outside $2R$ with multiplicity one.
- (c) The multiset $\mathcal{T} + \mathcal{T} = \{X + Y \mid X, Y \in \mathcal{T}\}$ contains the elements of $2R$ with multiplicity one, and half of the elements outside $2R$ with multiplicity two.
- (d) The multiset $\mathcal{T} + \mathcal{T}$ and $-(\mathcal{T} + \mathcal{T})$ coincide for m even. For m odd they intersect in $2R$. In particular, elements of $-\mathcal{T}$ occur in $\mathcal{T} + \mathcal{T}$ only for m even.

The next two lemmas are natural generalizations of Lemma B.2.

Lemma B.3 Let $R = GR(4, m)$, m odd, and \mathcal{T} be the Teichmüller set. Then

1. the multiset $\mathcal{T} + \mathcal{T} + \mathcal{T} = \{X + Y + Z \mid X, Y, Z \in \mathcal{T}\}$ contains each element of $-\mathcal{T}$ with multiplicity one, and the elements outside $-\mathcal{T}$ with multiplicity $2^m + 1$;

2. the multiset $\mathcal{T} + \mathcal{T} - \mathcal{T} = \{X + Y - Z \mid X, Y, Z \in \mathcal{T}\}$ contains each element of \mathcal{T} with multiplicity $2^{m+1} - 1$, and the elements outside \mathcal{T} with multiplicity $2^m - 1$.

Proof. Here we only give the proof of (1), since a similar argument will prove (2). We need to investigate the number of solutions of the following equation

$$X + Y + Z = C,$$

where $X, Y, Z \in \mathcal{T}$ and $C \in R$. The element C can be uniquely expressed as $C = A + 2B$, where $A, B \in \mathcal{T}$.

First if $C \in -\mathcal{T}$ i.e., $A = B$, then we have $X + Y + Z + A = 0$, implying $X = Y = Z = A$. So the multiset $\mathcal{T} + \mathcal{T} + \mathcal{T}$ contains each element of $-\mathcal{T}$ with multiplicity one.

Next if $C \notin -\mathcal{T}$, i.e., $A \neq B$, we split our consideration into two cases $A = 0$ and $A \neq 0$.

When $A = 0$, without loss of generality, we consider the equation

$$X + Y + Z = 2, \tag{6}$$

which is equivalent to the following system of binary equations

$$\begin{cases} x + y + z = 0, \\ xy + xz + yz = 1, \end{cases} \tag{7}$$

by Lemma B.1. From this system, we have

$$x^2 + y^2 + xy + 1 = 0.$$

Assume that $x = 0$, we get one solution

$$(x, y, z) = (0, 1, 1).$$

Assume that $x \neq 0$ and $y = tx$ for some $t \in \mathbb{F}_{2^m}$, then we obtain

$$(t^2 + t + 1)x^2 + 1 = 0,$$

implying

$$(x, y, z) = \left(\frac{1}{\sqrt{t^2 + t + 1}}, \frac{t}{\sqrt{t^2 + t + 1}}, \frac{t + 1}{\sqrt{t^2 + t + 1}} \right).$$

Hence there are totally $2^m + 1$ solutions for Equation (6).

When $A \neq 0$, without loss of generality, we only need to investigate the equation

$$X + Y + Z = 1 + 2B, \quad (8)$$

where $B \in \mathcal{T}$ and $B \neq 1$. By Lemma B.1, Equation (8) is equivalent to the following system

$$\begin{cases} x + y + z = 1, \\ xy + xz + yz = b^2, \end{cases} \quad (9)$$

from which we have

$$x^2 + xy + y^2 + x + y + b^2 = 0. \quad (10)$$

Now let $x = (1 + b)u + b$ and $y = (1 + b)v + b$. Then the above equation becomes

$$u^2 + uv + v^2 + u + v = 0,$$

so without loss of generality, we can assume $b = 0$ in Equation (10).

Assume that $x = 0$, then we get two solutions $(x, y, z) = (0, 0, 1)$ and $(x, y, z) = (0, 1, 0)$.

Assume that $x \neq 0$ and $y = tx$, for some $t \in \mathbb{F}_{2^m}$, then we have

$$(t^2 + t + 1)x^2 + (t + 1)x = 0$$

implying

$$(x, y, z) = (0, 0, 1)$$

or

$$(x, y, z) = \left(\frac{t + 1}{t^2 + t + 1}, \frac{t^2 + t}{t^2 + t + 1}, \frac{t}{t^2 + t + 1} \right).$$

If $t = 1$, there is only one solution $(x, y, z) = (0, 0, 1)$. Hence there are totally $2^m + 1$ solutions for Equation (10). \square

The above lemma implies the following result immediately.

Corollary B.1 *Let $R = GR(4, m)$, m odd, and \mathcal{T} be the Teichmüller set.*

- (a) *The multiset $\mathcal{T} + \mathcal{T} + \mathcal{T} + \mathcal{T} = \{X + Y + Z + W : X, Y, Z, W \in \mathcal{T}\}$ contains 0 with multiplicity 2^m , the elements of $2R \setminus \{0\}$ with multiplicity $2^m(2^m + 1)$, and the elements outside $2R$ with multiplicity 2^{2m} .*
- (b) *The multiset $\mathcal{T} + \mathcal{T} - \mathcal{T} - \mathcal{T} = \{X + Y - Z - W : X, Y, Z, W \in \mathcal{T}\}$ contains 0 with multiplicity $(2^{m+1} - 1)2^m$, the elements of $2R \setminus \{0\}$ with multiplicity $(2^m - 1)2^m$, and the elements outside $2R$ with multiplicity 2^{2m} .*

- (c) The multiset $\mathcal{T} + \mathcal{T} + \mathcal{T} - \mathcal{T} = \{X + Y + Z - W : X, Y, Z, W \in \mathcal{T}\}$ contains the elements of $2R$ with multiplicity 2^{2m} , the elements of $(\mathcal{T} + \mathcal{T}) \setminus 2R$ with multiplicity $(2^m + 1)2^m$, and the elements outside $\mathcal{T} + \mathcal{T}$ with multiplicity $(2^m - 1)2^m$.

Let $f_a(x) = x^3 + x + a$ and set

$$M_i = \{a \in \mathbb{F}_{2^m}, a \neq 0 \mid f_a(x) = 0 \text{ has precisely } i \text{ solutions in } \mathbb{F}_{2^m}\}$$

for $i = 0, 1, 3$. The exact values of the three numbers $|M_0|, |M_1|, |M_3|$ have been computed in the appendix of [16]:

$$\begin{aligned} |M_0| &= \frac{q+1}{3}, \\ |M_1| &= \frac{q}{2} - 1, \quad \text{and} \\ |M_3| &= \frac{q-2}{6}. \end{aligned}$$

Lemma B.4 *Let $f_a(x) = x^3 + x + a$ for some $a \in \mathbb{F}_{2^m}$.*

- (1) *If $a = 0$, then $f_a(x)$ has two zeroes $x = 0$ and 1 in \mathbb{F}_{2^m} .*
- (2) *If $a = b + b^{-1}$ for some $b \in \mathbb{F}_{2^m} \setminus \{0, 1\}$, then $f_a(x)$ has one and only one zero in \mathbb{F}_{2^m} .*
- (3) *If $a = b^{-1} + b^{-3}$ for some $b \in \mathbb{F}_{2^m} \setminus \{0, 1\}$ satisfying $\text{tr}(b) = 1$, then $f_a(x)$ has three distinct zeroes in \mathbb{F}_{2^m} .*
- (4) *If a satisfies none of the above conditions, then $f_a(x)$ is irreducible over \mathbb{F}_{2^m} .*

Proof. (1) is immediate. One finds in the literature (see [5, p.169]) that $f_a(x) = 0$ has a unique solution in \mathbb{F}_{2^m} if and only if $\text{tr}(1/a) = 0$. If $a = b + b^{-1}$ for some $b \in \mathbb{F}_{2^m} \setminus \{0, 1\}$, then

$$\text{tr}\left(\frac{1}{b + b^{-1}}\right) = \text{tr}\left(\frac{b}{b^2 + 1}\right) = \text{tr}\left(\frac{b}{b + 1} + \left(\frac{b}{b + 1}\right)^2\right) = 0.$$

It is easy to check that the size of the set $\{b + b^{-1} \mid b \in \mathbb{F}_{2^m} \setminus \{0, 1\}\}$ is equal to $|M_1|$. Then (2) follows.

If $a = b^{-1} + b^{-3}$ for some $b \in \mathbb{F}_{2^m} \setminus \{0, 1\}$ satisfying $\text{tr}(b) = 1$, then we have

$$\text{tr}\left(\frac{1}{b^{-1} + b^{-3}}\right) = \text{tr}\left(b + \frac{b}{b^2 + 1}\right) = \text{tr}(b) = 1.$$

Since b^{-1} is already a zero of $f_a(x)$, we see that $f_a(x)$ must have three distinct zeroes in \mathbb{F}_{2^m} . The next thing is to check that the cardinality of the set $\{b^{-1} + b^{-3} \mid b \in \mathbb{F}_{2^m} \setminus \{0, 1\}, \text{tr}(b) = 1\}$ is the same as $|M_3|$. Suppose that $b^{-1} + b^{-3} = c^{-1} + c^{-3}$ for some $b \in \mathbb{F}_{2^m} \setminus \{0, 1\}$ satisfying $\text{tr}(b) = 1$ and $c \in \mathbb{F}_{2^m} \setminus \{0, 1\}$. We see that

$$\begin{aligned} b^{-1} + b^{-3} &= c^{-1} + c^{-3} \\ \iff (b+c)(b^2c^2 + b^2 + bc + c^2) &= 0 \\ \iff c = b \text{ or } c^2 + (b/(b^2 + 1))c + b^2/(b^2 + 1) &= 0. \end{aligned}$$

Because

$$\text{tr}\left(\frac{b^2/(b^2 + 1)}{b^2/(b^2 + 1)^2}\right) = \text{tr}(b^2 + 1) = 0,$$

the equation $c^2 + (b/(b^2 + 1))c + b^2/(b^2 + 1) = 0$ has two distinct zeroes in \mathbb{F}_{2^m} . From the symmetry of the equation $b^2c^2 + b^2 + bc + c^2 = 0$, the value $\text{tr}(c)$ must equal to 1. Now it follows that the size of the set $\{b^{-1} + b^{-3} \mid b \in \mathbb{F}_{2^m} \setminus \{0, 1\}, \text{tr}(b) = 1\}$ is $(q - 2)/6 = |M_3|$, finishing the prove of (3). Then (4) is immediate. \square

Lemma B.5 *Let $W \in \mathcal{T}$. The number of solutions to the following system of equations*

$$\begin{cases} X + Y - Z = W + 2; \\ x^3 + y^3 + z^3 = w^3 + d, \end{cases}$$

is (1) 1, when $d = 0$; (2) 2, when $d \in M_1$; (3) 0, when $d \in M_0 \cup M_3$.

Proof. Using the fact that $X + Y = (\sqrt{X} + \sqrt{Y})^2 + 2\sqrt{XY}$, the first equation $X + Y = 2 + Z + W$ translates to the following two equations over \mathbb{F}_{2^m} :

$$x + y = z + w, \quad xy = zw + 1.$$

Now we compute

$$\begin{aligned} d &= x^3 + y^3 + z^3 + w^3 \\ &= (x + y)^3 + xy(x + y) + z^3 + w^3 \\ &= (z + w)^3 + (zw + 1)(z + w) + z^3 + w^3 \\ &= z + w. \end{aligned}$$

Hence $z = d + w$ and $y = d + x$. Plugging them into $xy = zw + 1$, we get $(x + z)^2 + d(x + z) + 1 = 0$. When $d = 0$, we get $x = 1 + z = d + w + 1$, so this system only has 1 solution. When $d \neq 0$, we have $(x + z)^2/d^2 + (x + z)/d + 1/d^2 = 0$. This equation has 0 or 2 solutions depending on whether $\text{tr}(1/d) = 1$ or not. \square

Lemma B.6 *Let $A, B \in \mathcal{T}$ and $B \neq 0$. The number of solutions to the following system of equations*

$$\begin{cases} X + Y - Z = A + 2B; \\ x^3 + y^3 + z^3 = a^3 + b^3d, \end{cases}$$

is (1) 1, when $d = 0$; (2) 2, when $d \in M_1$; (3) 0, when $d \in M_0 \cup M_3$.

Similarly, we get the following result.

Lemma B.7 *Let $A, B \in \mathcal{T}$ and $B \neq 0$. The number of solutions to the following system*

$$\begin{cases} X + Y + Z = -A + 2B, \\ x^3 + y^3 + z^3 = a^3 + b^3d, \end{cases}$$

is (1) 3, when $d = 0$; (2) 0, when $d \in M_0 \cup M_1$; (3) 6, when $d \in M_3$.

The following corollaries can be easily verified from the proof of the above lemmas.

Corollary B.2 *Let $0 \neq B \in \mathcal{T}$. The number of solutions to the following system of equations*

$$\begin{cases} X + Y - Z - W = 2B; \\ x^3 + y^3 + z^3 + w^3 = b^3d, \end{cases}$$

is (1) 2^m , when $d = 0$; (2) 2^{m+1} , when $d \in M_1$; (3) 0, when $d \in M_0 \cup M_3$.

Corollary B.3 *Let $0 \neq B \in \mathcal{T}$. The number of solutions to the following system*

$$\begin{cases} X + Y + Z + W = 2B; \\ x^3 + y^3 + z^3 + w^3 = b^3d, \end{cases}$$

is (1) $3 \cdot 2^m$, when $d = 0$; (2) 0, when $d \in M_0 \cup M_1$; (3) $6 \cdot 2^m$, when $d \in M_3$.

8 Appendix C

For $(a, b) \in G \times \mathbb{F}_q$, the sum $\xi(a, b)$ has the following properties:

1. If $b \neq 0$, then $\xi(a, b) = \xi(aB^{-1/3}, 1)$;
2. If $U, V \in \mathcal{T}, W \in \mathcal{T}^*$, then $\xi(U + 2V, b) = \xi(UW + 2VW, bw^3)$.

Now we fix some notation. Let a and c be elements of R , and write them as $a = U + 2V$ and $c = S + 2T$ where $U, V, S, T \in \mathcal{T}$. For convenience, we define η_a as $\eta_a = \xi(a, 1)$. Let u be the projection of a modulo 2 in \mathbb{F}_q . Set

$$f_u(z) = z^2 + u^2z + \sqrt{z} + u$$

and F_u be the zeros of $f_u(z)$ in \mathbb{F}_q . Also set

$$h_u(z) = f_u(z) - u = z^2 + u^2z + \sqrt{z}$$

and H_u be the zeros of $h_u(z)$ in \mathbb{F}_q . It is easy to see $u^2 \in F_u$ and

$$F_u = \{x + u^2 \mid x \in H_u\},$$

so $|F_u| = |H_u|$. For each $x \in H_u$, we have $\text{tr}(ux) = \text{tr}(u^2x^2) = \text{tr}(x^3 + x^{3/2}) = 0$, which yields that $\text{tr}(uy) = \text{tr}(u^3)$ for each $y \in F_u$. For $X, Y \in \mathcal{T}$, we see that

$$X + Y = (\sqrt{X} + \sqrt{Y})^2 + 2\sqrt{XY}.$$

The element $(\sqrt{X} + \sqrt{Y})^2 \in \mathcal{T}$ will be denoted as $X \oplus Y$ in the following. Here come some useful results involving η_a .

Lemma C.1 *Let $a \in R$ and u be the projection of a modulo 2 in \mathbb{F}_q . The exponential sum η_a satisfies:*

$$\begin{aligned} \eta_a^2 &= 2^m \sum_{\substack{Z \in \mathcal{T} \\ f_u(z)=0}} i^{\text{T}(aZ+2Z^3)}, & \eta_a \overline{\eta_a} &= 2^m \sum_{\substack{Z \in \mathcal{T} \\ h_u(z)=0}} i^{\text{T}(aZ+2Z^3)}, \\ \eta_a^4 &= 2^m (-1)^{\text{tr}(u^3)} |F_u| \eta_a \overline{\eta_a}, & (\eta_a \overline{\eta_a})^2 &= 2^m |F_u| \eta_a \overline{\eta_a}, & \eta_a^3 \overline{\eta_a} &= 2^m |F_u| \eta_a^2. \end{aligned}$$

Proof. We first compute

$$\begin{aligned} \eta_a^2 &= \sum_{X \in \mathcal{T}} \sum_{Y \in \mathcal{T}} i^{\text{T}(a(X+Y)+2(X^3+Y^3))} = \sum_{Y \in \mathcal{T}} \sum_{Z \in \mathcal{T}} i^{\text{T}(a(Y \oplus Z+Y)+2((Y \oplus Z)^3+Y^3))} \\ &= \sum_{Y \in \mathcal{T}} \sum_{Z \in \mathcal{T}} i^{\text{T}(aZ+2Z^3+2(aY+a\sqrt{Y}Z+Y^2Z+Z^2Y+Z^3))} \\ &= \sum_{Z \in \mathcal{T}} i^{\text{T}(aZ+2Z^3)} \sum_{y \in \mathbb{F}_q} (-1)^{\text{tr}(y(z^2+u^2z+\sqrt{z}+u))} \\ &= 2^m \sum_{\substack{Z \in \mathcal{T} \\ f_u(z)=0}} i^{\text{T}(aZ+2Z^3)}. \end{aligned}$$

A similar analysis would give

$$\eta_a \overline{\eta_a} = 2^m \sum_{\substack{Z \in \mathcal{T} \\ h_u(z)=0}} i^{\text{T}(aZ+2Z^3)}.$$

Then

$$\begin{aligned}
\eta_a^4 &= 2^{2m} \sum_{\substack{Z \in \mathcal{T} \\ f_u(z)=0}} \sum_{\substack{W \in \mathcal{T} \\ f_u(w)=0}} i^{\text{T}(a(Z+W)+2(Z^3+W^3))} \\
&= 2^{2m} \sum_{\substack{Z \in \mathcal{T} \\ f_u(z)=0}} \sum_{\substack{W \in \mathcal{T} \\ h_u(w)=0}} i^{\text{T}(a(Z+Z \oplus W)+2(Z^3+(Z \oplus W)^3))} \\
&= 2^{2m} \sum_{\substack{W \in \mathcal{T} \\ h_u(w)=0}} i^{\text{T}(aW+2W^3)} \sum_{\substack{Z \in \mathcal{T} \\ f_u(z)=0}} (-1)^{\text{tr}(uz+w(z^2+u^2z+\sqrt{z}))} \\
&= 2^{2m} \sum_{\substack{W \in \mathcal{T} \\ h_u(w)=0}} i^{\text{T}(aW+2W^3)} \sum_{\substack{Z \in \mathcal{T} \\ f_u(z)=0}} (-1)^{\text{tr}(u(z+w))} \\
&= 2^m (-1)^{\text{tr}(u^3)} |F_u| \eta_a \overline{\eta_a}.
\end{aligned}$$

The same calculations as above will prove the rest of equations. \square

Lemma C.2 *Let $c \in R \setminus 2R$ and $d \in \mathbb{F}_q$. Then c can be expressed uniquely as $c = F - G$ where $F, G \in \mathcal{T}$. We have*

$$\mathbf{E}(c, d) = \begin{cases} 2^{3m+4}(3 \cdot 2^{m-1} - 1), & \text{if } d = f^3 + g^3; \\ 2^{3m+4}(2^{m-1} - 1), & \text{if } d \neq f^3 + g^3, \end{cases}$$

where f, g are the projections of F, G modulo 2 in \mathbb{F}_q .

Proof. It is direct to check that

$$st^2 + s^{-1}t^4 + s^3 = f^3 + g^3$$

and

$$\text{tr}(s^{-3}(f^3 + g^3)) = 1.$$

Let $X \in \mathcal{T}$ and $B \in \mathcal{T}^*$. We first compute that

$$\begin{aligned}
\mathcal{U} &:= i^{\text{T}(Xc)} \sum_{Y \in \mathcal{T}} \eta_{(X+2Y)B^{-1/3}} \overline{\eta_{(X+2Y)B^{-1/3}}} (-1)^{\text{tr}(ys)} \\
&= 2^m i^{\text{T}(Xc)} \sum_{Y \in \mathcal{T}} \sum_{\substack{Z \in \mathcal{T} \\ h_{x'}(z)=0}} i^{\text{T}((X+2Y)B^{-1/3})Z+2Z^3} (-1)^{\text{tr}(ys)} \\
&= 2^m i^{\text{T}(Xc)} \sum_{\substack{Z \in \mathcal{T} \\ h_{x'}(z)=0}} i^{\text{T}(XB^{-1/3})Z+2Z^3} \sum_{Y \in \mathcal{T}} (-1)^{\text{tr}(y(b^{-1/3}z+s))} \\
&= \begin{cases} 2^{2m} (-1)^{\text{tr}(b(f^3+g^3))}, & \text{if } X = B^{\frac{1}{2}} S^{\frac{1}{2}} \oplus B^{\frac{1}{4}} S^{-\frac{1}{4}}; \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

where $x' = xb^{-1/3}$. Notice that $B^{\frac{1}{2}}S^{\frac{1}{2}} \oplus B^{\frac{1}{4}}S^{-\frac{1}{4}} = 0$ if $b = s^{-3}$.

Let $x, s \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q^* \setminus \{s^{-3}\}$. We consider the set H_x . Immediately 0 is a root of $h_x(z)$. The equation $h_x(z) = 0$ is equivalent to

$$(h_x(z))^2 = z(z^3 + x^4z + 1) = 0.$$

Replace z with x^2w , then $z^3 + x^4z + 1 = 0$ becomes $w^3 + w + x^{-6} = 0$.

If

$$x = (bs)^{1/2} + (bs^{-1})^{1/4},$$

then

$$x^6 = b^{1/2}s^{3/2} + bs^3 + b^{-1/2}s^{-3/2} + 1$$

and

$$x^{-6} = (1 + b^{-1/2}s^{-3/2})^{-1} + (1 + b^{-1/2}s^{-3/2})^{-3}.$$

Using Lemma B.4, we see that if

$$\text{tr}(1 + b^{-1/2}s^{-3/2}) = 0,$$

then

$$\text{tr}(x^6) = \text{tr}(1 + b^{-1}s^{-3}) = 0,$$

implying

$$|F_{b^{\frac{1}{6}}s^{\frac{1}{2}}+b^{-\frac{1}{12}}s^{-\frac{1}{4}}}| = |H_{b^{\frac{1}{6}}s^{\frac{1}{2}}+b^{-\frac{1}{12}}s^{-\frac{1}{4}}}| = 2;$$

otherwise we have

$$|F_{b^{\frac{1}{6}}s^{\frac{1}{2}}+b^{-\frac{1}{12}}s^{-\frac{1}{4}}}| = |H_{b^{\frac{1}{6}}s^{\frac{1}{2}}+b^{-\frac{1}{12}}s^{-\frac{1}{4}}}| = 4.$$

So the expression

$$(2(-1)^{\text{tr}(1+b^{-1}s^{-3})} + 6)|F_{b^{\frac{1}{6}}s^{\frac{1}{2}}+b^{-\frac{1}{12}}s^{-\frac{1}{4}}}|$$

is always equal to 16.

The sum $\mathbf{E}(c, d)$ will be divided into four parts, and computed separately with the assistance of Lemma C.1. We first compute

$$\begin{aligned}
\mathbf{E}(c, d)_1 &= \sum_{a \in R^*} \sum_{b \in \mathbb{F}_q^*} \left(\xi^4(a, b) + \overline{\xi^4(a, b)} + 6 \xi^2(a, b) \overline{\xi^2(a, b)} \right) i^{\mathrm{T}(ac+2bd)} \\
&= \sum_{X \in \mathcal{T}^*} \sum_{Y \in \mathcal{T}} \sum_{b \in \mathbb{F}_q^*} \left(\xi^4(X + 2Y, b) + \overline{\xi^4(X + 2Y, b)} \right. \\
&\quad \left. + 6 \xi^2(X + 2Y, b) \overline{\xi^2(X + 2Y, b)} \right) i^{\mathrm{T}((X+2Y)c+2bd)} \\
&= \sum_{X \in \mathcal{T}^*} \sum_{Y \in \mathcal{T}} \sum_{b \in \mathbb{F}_q^*} \left(\eta_{(X+2Y)B^{-\frac{1}{3}}}^4 + \overline{\eta_{(X+2Y)B^{-\frac{1}{3}}}^4} \right. \\
&\quad \left. + 6 \eta_{(X+2Y)B^{-\frac{1}{3}}}^2 \overline{\eta_{(X+2Y)B^{-\frac{1}{3}}}^2} \right) i^{\mathrm{T}((X+2Y)c+2bd)} \\
&= 2^m \sum_{X \in \mathcal{T}^*} \sum_{b \in \mathbb{F}_q^*} (-1)^{\mathrm{tr}(bd)} \left(2(-1)^{\mathrm{tr}(x^3 b^{-1})} + 6 \right) |F_{xb^{-\frac{1}{3}}}| \mathcal{U} \\
&= 2^{3m} \sum_{b \in \mathbb{F}_q^* \setminus \{s^{-3}\}} \left[\left(2(-1)^{\mathrm{tr}(1+b^{-1}s^{-3})} + 6 \right) |F_{b^{\frac{1}{6}} s^{\frac{1}{2}} + b^{-\frac{1}{12}} s^{-\frac{1}{4}}}| \right] (-1)^{\mathrm{tr}(b(f^3+g^3+d))} \\
&= 2^{3m+4} \sum_{b \in \mathbb{F}_q^* \setminus \{s^{-3}\}} (-1)^{\mathrm{tr}(b(f^3+g^3+d))}.
\end{aligned}$$

Next,

$$\begin{aligned}
\mathbf{E}(c, d)_2 &= \sum_{X \in \mathcal{T}^*} \sum_{Y \in \mathcal{T}} \left(\xi^4(X + 2Y, 0) + \overline{\xi^4(X + 2Y, 0)} \right. \\
&\quad \left. + 6 \xi^2(X + 2Y, 0) \overline{\xi^2(X + 2Y, 0)} \right) i^{\mathrm{T}((X+2Y)c)} \\
&= \sum_{X \in \mathcal{T}^*} \sum_{Y \in \mathcal{T}} \left(2^{2m} i^{\mathrm{T}(2)} + 2^{2m} i^{-\mathrm{T}(2)} + 6 \cdot 2^{2m} \right) i^{\mathrm{T}((X+2Y)c)} \\
&= 2^{2m+2} \sum_{X \in \mathcal{T}^*} i^{\mathrm{T}(Xc)} \sum_{Y \in \mathcal{T}} (-1)^{\mathrm{tr}(ys)} \\
&= 0.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbf{E}(c, d)_3 &= \sum_{Y \in \mathcal{T}} \sum_{b \in \mathbb{F}_q^*} \left(\xi^4(2Y, b) + \overline{\xi^4(2Y, b)} + 6\xi^2(2Y, b) \overline{\xi^2(2Y, b)} \right) i^{\text{T}(2Yc+2bd)} \\
&= \sum_{Y \in \mathcal{T}} \sum_{b \in \mathbb{F}_q^*} \left(2^{2m+4} (1 + (-1)^{\text{tr}(yb^{-1/3}+1)}) \right) (-1)^{\text{tr}(ys+bd)} \\
&= 2^{2m+4} \sum_{b \in \mathbb{F}_q^*} (-1)^{\text{tr}(bd+1)} \sum_{Y \in \mathcal{T}} (-1)^{\text{tr}(y(b^{-1/3}+s))} \\
&= 2^{3m+4} (-1)^{\text{tr}(s^{-3}d+1)}.
\end{aligned}$$

At last,

$$\mathbf{E}(c, d)_4 = \xi^4(0, 0) + \overline{\xi^4(0, 0)} + 6\xi^2(0, 0) \overline{\xi^2(0, 0)} = 2^{4m+3}.$$

Adding $\mathbf{E}(c, d)_1, \mathbf{E}(c, d)_2, \mathbf{E}(c, d)_3$, and $\mathbf{E}(c, d)_4$ up will complete the proof. \square

Lemma C.3 *Let $c \in R \setminus 2R$ and $d \in \mathbb{F}_q$. There exist $F, G \in \mathcal{T}$ such that $c = F + G$ with $F \neq G$. We have*

$$\mathbf{F}(c, d) = \begin{cases} 2^{3m+2}(3 \cdot 2^{m-1} - 1), & \text{if } d = f^3 + g^3; \\ 2^{3m+2}(2^{m-1} - 1), & \text{if } d \neq f^3 + g^3, \end{cases}$$

where f, g are the projections of F, G modulo 2 in \mathbb{F}_q .

Proof. It is easy to check that

$$st^2 + s^3 = f^3 + g^3.$$

Let $X \in \mathcal{T}$ and $B \in \mathcal{T}^*$. We see

$$\begin{aligned}
\mathcal{V} &:= i^{\text{T}(X)} \sum_{Y \in \mathcal{T}} \left(\eta_{(X+2Y)B^{-1/3}}^3 \overline{\eta_{(X+2Y)B^{-1/3}}} \right. \\
&\quad \left. + \eta_{(X+2Y)B^{-1/3}} \overline{\eta_{(X+2Y)B^{-1/3}}^3} \right) (-1)^{\text{tr}(y)} \\
&= 2^{2m} |K_{xb^{-1/3}}| i^{\text{T}(X)} \sum_{\substack{Z \in \mathcal{T} \\ f_{x'}(z)=0}} \left(i^{\text{T}(XB^{-1/3}Z+2Z^3)} \right. \\
&\quad \left. + i^{-\text{T}(XB^{-1/3}Z+2Z^3)} \right) \sum_{Y \in \mathcal{T}} (-1)^{\text{tr}(v(b^{-1/3}z+1))} \\
&= \begin{cases} 2^{3m} |K_{b^{1/6}}| (1 + (-1)^{\text{tr}(b)}), & \text{if } U = B^{1/2}; \\ 2^{3m} |K_{b^{1/6}+b^{-1/3}}| (-1 + (-1)^{\text{tr}(b)}), & \text{if } U = 1 \oplus B^{1/2}; \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

where $x' = xb^{-1/3}$.

We first calculate

$$\begin{aligned}
\mathbf{F}(c, d)_1 &= \sum_{X \in \mathcal{T}^*} \sum_{Y \in \mathcal{T}} \sum_{b \in \mathbb{F}_q^*} \left(\xi^3(X + 2Y, b) \overline{\xi(X + 2Y, b)} \right. \\
&\quad \left. + \xi(X + 2Y, b) \overline{\xi^3(X + 2Y, b)} \right) i^{\mathrm{T}((X+2Y)c+2bd)} \\
&= \sum_{X \in \mathcal{T}^*} \sum_{Y \in \mathcal{T}} \sum_{b \in \mathbb{F}_q^*} \left(\xi^3((X + 2Y)S, bs^3) \overline{\xi((X + 2Y)S, bs^3)} \right. \\
&\quad \left. + \xi((X + 2Y)S, bs^3) \overline{\xi^3((X + 2Y)S, bs^3)} \right) i^{\mathrm{T}((X+2Y)S(1+2S^{-1}T)+2bd)} \\
&= \sum_{X \in \mathcal{T}^*} \sum_{Y \in \mathcal{T}} \sum_{b \in \mathbb{F}_q^*} \left(\xi^3(X + 2Y, b) \overline{\xi(X + 2Y, b)} \right. \\
&\quad \left. + \xi(X + 2Y, b) \overline{\xi^3(X + 2Y, b)} \right) i^{\mathrm{T}((X+2Y)(1+2S^{-1}T)+2bs^{-3}d)} \\
&= \sum_{X \in \mathcal{T}^*} \sum_{b \in \mathbb{F}_q^*} i^{\mathrm{T}(2XS^{-1}T+2bs^{-3}d)} \mathcal{V} \\
&= 2^{3m} \sum_{b \in \mathbb{F}_q^*} |K_{b^{1/6}}| (-1)^{\mathrm{tr}(b^{1/2}s^{-1}t+bs^{-3}d)} (1 + (-1)^{\mathrm{tr}(b)}) \\
&\quad + 2^{3m} \sum_{b \in \mathbb{F}_q \setminus \{0,1\}} |K_{b^{1/6}+b^{-1/3}}| (-1)^{\mathrm{tr}(b^{1/2}s^{-1}t+s^{-1}t+bs^{-3}d)} (-1 + (-1)^{\mathrm{tr}(b)}) \\
&= 2^{3m+2} \left(\sum_{\substack{b \in \mathbb{F}_q^* \\ \mathrm{tr}(b)=0}} (-1)^{\mathrm{tr}(b(s^{-2}t^2+s^{-3}d))} + \sum_{\substack{b \in \mathbb{F}_q \setminus \{0,1\} \\ \mathrm{tr}(b)=1}} (-1)^{\mathrm{tr}(s^{-1}t+b(s^{-2}t^2+s^{-3}d))} \right) \\
&= 2^{3m+2} (1 + (-1)^{\mathrm{tr}(s^{-3}d+1)}) \sum_{\substack{b \in \mathbb{F}_q^* \\ \mathrm{tr}(b)=0}} (-1)^{\mathrm{tr}(b(s^{-2}t^2+s^{-3}d))} \\
&= \begin{cases} 2^{3m+3}(2^{m-1} - 1), & \text{if } d = f^3 + g^3; \\ -2^{3m+2}(1 + (-1)^{\mathrm{tr}(s^{-3}d+1)}), & \text{otherwise.} \end{cases}
\end{aligned}$$

Next,

$$\begin{aligned}
\mathbf{F}(c, d)_2 &= \sum_{X \in \mathcal{T}^*} \sum_{Y \in \mathcal{T}} \left(\xi^3(X + 2Y, 0) \overline{\xi(X + 2Y, 0)} \right. \\
&\quad \left. + \xi(X + 2Y, 0) \overline{\xi^3(X + 2Y, 0)} \right) i^{\text{T}((X+2Y)c)} \\
&= 2^{2m} \sum_{X \in \mathcal{T}^*} \sum_{Y \in \mathcal{T}} \left(i^{\text{T}(1+2X^{-1}Y+(X+2Y)c)} + i^{\text{T}(-1-2X^{-1}Y+(X+2Y)c)} \right) \\
&= 2^{2m} \left(\sum_{X \in \mathcal{T}^*} i^{\text{T}(1+Xc)} + \sum_{X \in \mathcal{T}^*} i^{\text{T}(-1+Xc)} \right) \sum_{Y \in \mathcal{T}} (-1)^{\text{tr}(y(x^{-1}+s))} \\
&= 2^{3m} \left(i^{\text{T}(2+2s^{-1}t)} + i^{\text{T}(2s^{-1}t)} \right) \\
&= 0.
\end{aligned}$$

Then we calculate

$$\begin{aligned}
\mathbf{F}(c, d)_3 &= \sum_{Y \in \mathcal{T}} \sum_{b \in \mathbb{F}_q^*} (\xi^3(2Y, b) \overline{\xi(2Y, b)} + \xi(2Y, b) \overline{\xi^3(2Y, b)}) i^{\text{T}(2Yc+2bd)} \\
&= 2^{2m+2} \sum_{Y \in \mathcal{T}} \sum_{b \in \mathbb{F}_q^*} (1 + (-1)^{\text{tr}(yb^{-1/3}+1)}) (-1)^{\text{tr}(ys+bd)} \\
&= 2^{2m+2} \sum_{b \in \mathbb{F}_q^*} (-1)^{\text{tr}(bd+1)} \sum_{Y \in \mathcal{T}} (-1)^{\text{tr}(y(b^{-1/3}+s))} \\
&= 2^{3m+2} (-1)^{\text{tr}(s^{-3}d+1)}
\end{aligned}$$

At last,

$$\mathbf{F}(c, d)_4 = \xi^3(0, 0) \overline{\xi(0, 0)} + \xi(0, 0) \overline{\xi^3(0, 0)} = 2^{4m+1}.$$

Adding $\mathbf{F}(c, d)_1$, $\mathbf{F}(c, d)_2$, $\mathbf{F}(c, d)_3$, and $\mathbf{F}(c, d)_4$ up will complete the proof. □

The following result can be proved similarly as above.

Lemma C.4 *Let $c \in 2R$ and $d \in \mathbb{F}_q$. We have*

$$\mathbf{F}(c, d) = \begin{cases} 2^{3m+2}(3 \cdot 2^{m-1} - 1), & \text{if } d = 0; \\ 2^{3m+2}(2^{m-1} - 1), & \text{if } d \neq 0. \end{cases}$$

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